Differential Geometry and General Relativity

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Introduction

TBD

1 Tensors

Tensors and tensor calculus are the main tool used in Differential Geometry and General Relativity. It is essential to understand how they work and how to use them.

1.1 Cartesian Tensors

Tensors fields are an extension of the idea of functions (i.e scalars), vector fields and a matrices with smooth function coefficients. They come with a rank.

Example: *Temperature in a room. A function on an open subset of $\mathbb{R}^n$ is a rank 0 tensor field (field means that its value depends on the point in the open set). It is commonly called a scalar.*

Example: *Velocity field in a fluid. A vector field on an open subset of $\mathbb{R}^n$ is a rank 1 tensor field (field means that its value depends on the point in the open set).*

NOTE: A vector field V can be seen as a linear function from V(w) defined as follows:

$$\mathbf{V}(\mathbf{w}) : \mathbf{w} \in \mathbb{R}^n \to \mathbb{R}; \;\; \mathbf{V}(\mathbf{w}) := \mathbf{V} \cdot \mathbf{w} \tag{1}$$

where the dot represents the scalar product. It is easy to see that the above function is linear. This is an important point which is always good to keep in mind. Vectors are always linear functions from $\mathbb{R}^n to \mathbb{R}$. This is a property they have in common with directional derivatives. We will use this later on.
Example: *Stress Tensor field . Stress is force per unit area. However to fully define stress in a linear elastic material we need a bit more data then a force in a point. This is because we have normal and tangent forces to any surface in the material and passing through any point.*

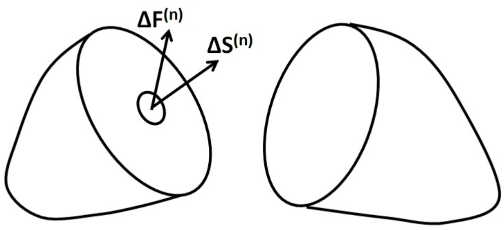

Figure 1: Internal Force in a Material

Consider a point P in the material and a flat element of surface $\Delta\boldsymbol{S}^{(n)}$, which orientation is given by its normal unit vector $\boldsymbol{n}$. The total internal contact force $\Delta\boldsymbol{F}^{(n)}$ on the surface (i.e. the force experienced by the material between the two faces of the surface) increases with the area of the surface and depends in amplitude and direction by $\boldsymbol{n}$. We define the following vector field depending on $\boldsymbol{n}$:

$$\boldsymbol{T}^{(n)} = \lim_{|\Delta\boldsymbol{S}^{(n)}| \to 0} \frac{\Delta\boldsymbol{F}^{(n)}}{|\Delta S^{(n)}|} \tag{2}$$

which is called Cauchy Traction Field and it has units of pressure. By using a reasoning which is not important to us and that is based on the equilibrium in the material (i.e. the internal of the material is still), it is possible to show that the relation between $\boldsymbol{T}^{(n)}$ and $\boldsymbol{n}$ is a linear transformation which is:

$$\boldsymbol{T}^{(n)} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \boldsymbol{T} \cdot \boldsymbol{n} \tag{3}$$

where $\boldsymbol{T}$ *is called the Cauchy Stress Tensor.*

NOTE: The Cauchy Traction Field $\mathbf{T}^{(n)}$ is a vector and therefore, $\mathbf{T}^{(n)}(\mathbf{m})$, with $\mathbf{m} \in \mathbb{R}^3$ a unit vector, is a linear map defined as:

$$\mathbf{T}^{(n)}(\mathbf{m}) : \mathbf{m} \in \mathbb{R}^3 \to \mathbb{R}; \;\; \mathbf{T}^{(n)}(\mathbf{m}) := \mathbf{T}^{(n)} \cdot \mathbf{m} \tag{4}$$

and $\mathbf{T}$ can therefore be seen as a bilinear map $\mathbf{T} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (in this case a quadratic form):

$$\mathbf{T}(\mathbf{n}; \mathbf{m}) : (\mathbf{n}; \mathbf{m}) \in \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}; \;\; \mathbf{T}(\mathbf{n}, \mathbf{m}) := \mathbf{T}^{(n)} \cdot \mathbf{m} = \mathbf{n}^{\intercal}\mathbf{T}\mathbf{m} \tag{5}$$

If $\mathbf{n}$ and $\mathbf{m}$ are unit vectors then $\mathbf{T}(\mathbf{n}; \mathbf{m})$ represents the projection along $\mathbf{m}$ of the Cauchy Stress Tensor and it has unit of pressure. However, in the above bilinear form we can use vectors of any amplitude.

Given the three examples above, the reader may have already guessed where we are getting at. We are not in a position for giving a proper definition of tensors yet because the tensors described in this section, called Cartesian tensors, are not the most general class of tensors. However, a tensor T of rank ν defined on an open subset $A \in \mathbb{R}^n$, is a multilinear map (i.e. linear in each of its ν entries) that at list for this class of tensors is of the form:

$$\mathbf{T} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R} \tag{6}$$

and it has n^ν components each of which is a smooth functions on A.

1.2 Covariant and Controvariant Tensors

So far we have introduced tensors on subset of $\mathbb{R}^n$ (i.e. an euclidean space) with components on constant canonical orthonormal basis. These tensors are called Cartesian Tensors. That is nice, however, if you want to do differential geometry and gravitation they are pretty much useless. The real power of tensors come into play when we deal with coordinates transformations. We will see that in this case, vectors can transform in two possible different ways. We show this with a simple example. Let us consider the good old differential of metavariable calculus for a function $f(x_1; x_2)$:

$$df(x) = \nabla f(x) \cdot dx; \;\; dx = (dx_1, dx_2) \tag{7}$$

the above differential, if we fix dx, is a scalar function of space (rank 0 tensor) and it is the scalar product of two vectors (rank 1 tensors). We apply now a generic coordinates transformation (i.e. we go to curvilinear coordinates) as follows:

$$\begin{cases} x_1 &= x_1(y_1, y_2) \\ x_2 &= x_2(y_1, y_2) \end{cases} \tag{8}$$

and in a neighbourhood of a point P where the above are invertible:

$$\begin{cases} y_1 &= y_1(x_1, x_2) \\ y_2 &= y_2(x_1, x_2) \end{cases} \tag{9}$$

and we want to evaluate:

$$df(y) = \nabla f(y) \cdot dy \tag{10}$$

Applying the chain rule to $\nabla f(x)$ and using (8) we find easily:

$$\nabla f(y) = \left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_2} \; , \; \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_2} \right) \tag{11}$$

and differentiating the (9) we have:

$$dy = \left(\frac{\partial y_1}{\partial x_1}dx_1 + \frac{\partial y_1}{\partial x_2}dx_2 \; , \; \frac{\partial y_2}{\partial x_1}dx_1 + \frac{\partial y_2}{\partial x_2}dx_2 \right) \tag{12}$$

clearly $\nabla f(y)$ and dy are both rank 1 tensors (i.e. vectors) but the transform in a completely different way. To better understand the difference we may use, as an example, a simple coordinate transformation such as a scaling, which is $x_1 = ay_1$ and $x_2 = ay_2$. Substituting in the (11) and (12) we get:

$$\nabla f(y) = a\,\nabla f(x); \;\; dy = \frac{1}{a}dx \tag{13}$$

while the scalar $df = \nabla f(y) \cdot dy$ remains invariant under the coordinate transformations.

- We will call all tensor transforming like (11) covariant vectors (or simply vectors).
- We will call all tensor transforming like (12) controvariant vectors (or covectors).

To sum up we have learned a few things some of which not explicitly said yet. Under coordinate transformations:

- Scalars are invariant. This is the reason why we defined tensors as multilinear maps with ν inputs of the form $\mathbf{T} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$. This maps are scalars and therefore invariant under coordinate transformations.
- Vectors can transform as covariant and controvariant vectors. In tensor calculus, we need to handle efficiently the way covariant and controvariant vector transform and this will be done using, for each point of the space where the tensor is defined, two separate set of bases for the two type of vectors.
- We need a different notation for covariant and controvariant tensors. This will be addressed later.
- If you think about it, we want a definition of scalar product which is invariant under coordinate transformations. This means that in the context of vector calculus scalar product makes sense only between a covariant and a controvariant vector.

We note finally that most of the equations in differential geometry like (11) and (12) derive directly from the chain rule and more in general from the rules on derivatives. Exaggerating we may say that in a way, differential geometry is about the chain rules and its effect on equations.

1.3 Dual Vector Spaces

We want to find an efficient way to handle how covariant and controvariant vectors change under coordinates transformations. We will do this by using two different coordinate basis, one for the components of the covariant vectors and one for the components of the controvariant vectors.

Let V be an n-dimensional vector space. We decide that all vectors (i.e. covariant vectors) are element of V. We find a base (any) of n vectors $(\mathbf{e}_1 \cdots, \mathbf{e}_n)$ and for each vector $\mathbf{v} \in V$ we represent the vector in an unique way as:

$$\mathbf{v} = \sum_1^n v^i \mathbf{e}_i \tag{14}$$

of its n components.

Notation: *We will use a down index for base vectors and an up index for a component of a vector.*

Notation: *We will use the Einstein Notation which is that every time we have an repeated index (one up and one down) in an equation, the summation is implied. For example we have:*

$$\boldsymbol{v} = \sum_1^n v^i \boldsymbol{e}_i = v^i \boldsymbol{e}_i \tag{15}$$

this will save use a lot of time writing summations.

Now let V^* be a vector space. We decide that all covectors (i.e. controvariant vectors) are element of V^*. We choose for V^* a base of n covectors $(\mathbf{e}^1 \cdot \cdot\cdot, \mathbf{e}^n)$ with the following requirement:

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq \text{j} \end{cases} \tag{16}$$

The reader should convince himself that when the base $\mathbf{e}_i$ is given for V, this induces an unique base $\mathbf{e}^i$ for V^*.

Notation: *We will use an up index for base covectors and a down index for a component of a covector.*

Definition: *We will call the vector space V^* defined above, the dual of the vector space V.*

Why we want to define dual vector spaces for covectors? Well, for example we have seen before that we want the scalar product to be invariant under coordinate transformations (i.e. to be a proper scalar tensor) and therefore we want to make legal only scalar product between a vector and a covector. If $\mathbf{v}$ is a vector and $\mathbf{w}$ is a covector, we have:

$$\mathbf{v} \cdot \mathbf{w} = v^i \mathbf{e}_i \cdot w_j \mathbf{e}^j = v^i w_i \mathbf{e}_j \cdot \mathbf{e}^j \tag{17}$$

because scalar product is commutative and associative. Given (16) we have eventually:

$$\mathbf{v} \cdot \mathbf{w} = v^i w_i \tag{18}$$

which means that we can apply the usual definition of scalar product to components while the problems related to how vectors and covectors change under coordinate transformations are handled by the way the relevant basis change. In tensor calculus, this will apply to components of tensors in the same way and it will make our life much simpler.

There is a one to one map between vector in V and covectors in V^*. This is why we call them a dual spaces (V is in turn the dual of V^*). It is like $\mathbf{v} \in V$ with components v^i and $\mathbf{v} \in V^*$ with components v_i are the same mathematical object but represented in a different way. The way components change to go from a vector to its dual covector and viceversa will be clear later on. However, you should know for the moment that exist two rank 2 tensors g_{ij} and g^{ij} that can be used to convert vectors in their dual covectors and viceversa as follows:

$$v^i = g^{ij} v_j \text{ and } v_i = g_{ij} v^j \tag{19}$$

The above operation is called lowering and rising indices. g_{ij} and g^{ij} are the dual of each other. g_{ij} and g^{ij} are a rank 2 tensor, the former with two covariant indices, the latter with 2 controvariant indices, a notation that you have not seen yet and that will be explained later. However, applying the definition of the Einstein notation given above, the meaning of (19) should be clear. If not, have a look two sections below where I explain Tensor algebra.

The tensors g_{ij} and g^{ij} are called the metric tensor. We will meet them later on and we will see that they have a very profound meaning and great importance in differential geometry.

1.4 Definition of a Tensor

TBD

1.5 Tensor Algebra

TBD

1.6 Coordinate Basis and Transformation Lows

TBD

1.7 Invariance of Tensor Equations

TBD

2 Differential Manifolds and Fibre Bundles

TBD

3 The metric Tensor

TBD

4 Connections and Covariant Derivatives

TBD

5 Geodesics

TBD

6 Curvature

TBD

7 Special Relativity

TBD

8 Einstein Field Equation

TBD

9 The Schwarzschild Solution

TBD

10 Gravitational Waves

TBD

11 Cosmology

TBD